Eur. Phys. J. C 7, 205–215 (1999)
DOI 10.1007/s100529801007

THE EUROPEAN PHYSICAL JOURNAL C

P-odd observables at the Υ peak

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Received: 14 April 1998 / Revised version: 25 August 1998 / Published online: 3 December 1998

Abstract. We study the γ-Z interference in the process $e^+e^- \to \Upsilon \to \tau^+\tau^-$ as a means to measure the neutral current coupling of the b-quark. The helicity amplitudes are calculated from resonant and background diagrams and the spin density matrix of the final state is discussed. The spin analyzer of the τ's is illustrated with the decays $\pi\nu$ and $\rho\nu \to (\pi\pi)\nu$. With $10^8 \Upsilon$ a sensitivity to g_V^b of a few per cent could be reachable.

1 Introduction

A precise determination of the fermionic electroweak couplings can provide strong hints on the nature of new physics at high scales through the quantum corrections to the effective theory at low energies. At present the excellent experimental agreement with the predictions of the Standard Model [1] is used to set bounds on this new dynamics beyond the Standard Model.

In this context, the agreement between the measured $Z - f\bar{f}$ couplings and the Standard Model is very good. However, while the lepton couplings have been independently measured for the three families at LEP, for the quarks only the b and c events can be separated in the hadronic event sample and consequently their couplings measured. From the measurements of R_b and A_b [2] we can get the values of the vector and axial couplings of Z to $b\bar{b}$ [3]. Using the results quoted in [2], and defining the vector and axial couplings in the Standard Model as $g_V^b = -1/4 + \sin^2\theta_W/3$ and $g_A^b = 1/4$, the accuracy on the bottom-Z couplings is: $\delta g_V^b = \pm 0.013$ and $\delta g_A^b = \pm 0.007$. It is important to notice that while the uncertainty in the g_A^b coupling is only of 2.8%, the g_V^b measurement has an uncertainty of 7.5%.

For the light quarks this separation has not been achieved at LEP and so an exclusive determination of their couplings is not available. A study of the final state distributions at different meson facilities can provide independent measurements of these couplings. In a previous work [4], we showed that a Φ factory with polarized e^- beams can supply the information on the s-quark couplings.

We show in this work that a detailed study of the final state distributions of the decay products of the τ's for $e^+e^- \to \Upsilon \to \tau^+\tau^-$ would provide valuable information on the vector $Z - b\bar{b}$ coupling, g_V^b.

To determine this coupling at energies well below the Z pole, we propose to measure the interference between γ and Z. The $b\bar{b}$ mesons which can couple to both γ and Z are the Υ mesons. We are interested in a process in which the Υ is coupled to a Z either in its production or in its decay. This means that our study for the final τ´s includes the decay of a polarized Υ or its weak decay.

For these reasons, we analyze the leptonic decays of the Υ resonances, therefore we could use $\Upsilon(1S)$, $\Upsilon(2S)$ and $\Upsilon(3S)$, but not $\Upsilon(4S)$ that decays dominantly to $B\bar{B}$ where the information on the Υ polarization is lost. In this work we will concentrate on the $\Upsilon(1S)$ resonance, but everything would be similar for $\Upsilon(2S)$ and $\Upsilon(3S)$. We can see in Table 1 that the branching ratios of $\Upsilon(1S)$ to the three charged leptons are approximately 2.5%, but e^+e^- and $\mu^+\mu^-$ can not be used because their polarizations are not measurable in the detector through decay distributions. Obviously e^+e^- are stable particles and $\mu^+\mu^-$ at this energy do not decay inside the detector. As we see explicitly below, all the relevant information on g_V^b that comes from a P-odd $\gamma - Z$ interference, appears at leading order in the polarization of the final leptons. This means that we are constrained to consider the decay $\Upsilon \to \tau^+\tau^-$ and to measure the τ polarizations.

To estimate the sensitivity of this process to the vector coupling, let us first consider the τ^- longitudinal polarization, suggested in [5] in another context. We make a reasonable approximation in order to get a simple and clean result: the resonant diagrams dominate the process on the $\Upsilon(1S)$ peak. So, we consider only diagrams (2), (3) and (4) in Fig. 1. Under these conditions we get a longitudinal polarization from the parity violating interferences of the dominant amplitude (2) with the neutral current amplitudes (3) + (4).

$$P_{z'} = 2\,\frac{8G_F}{\sqrt{2}}\,\frac{s}{4\pi\alpha}\,g_A\,\frac{g_V^b}{Q_b} \times \frac{(1+\cos^2\theta)|\mathbf{p}|\;p^0 + 2\cos\theta (p^0)^2}{(1+\cos^2\theta)(p^0)^2 + \sin^2\theta M_\tau^2} \tag{1}$$

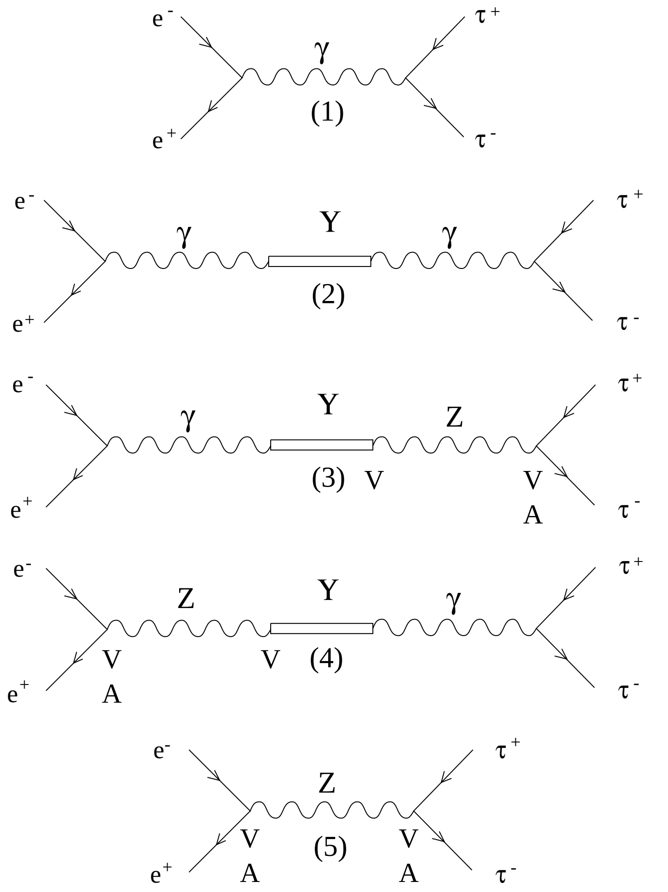

Fig. 1. Resonant and non-resonant diagrams for the process $e^+e^- \to \tau^+\tau^-$ around the Υ resonance

where $g_A = 1/4$ is the axial coupling to the leptons, Q_b the charge of the b quark and $p^\mu = (p^0, \mathbf{p}(\theta))$ the four-momentum of the τ^-.

In this expression we can discover some interesting features of this observable:

- It is linear on g_V^b, the vector coupling which we want to determine, showing up together with the axial coupling of the Z to leptons.
- The magnitude of the τ-polarization on the Υ peak is set by the factor $8G_F M_\Upsilon^2/(4\pi\alpha\sqrt{2}) \simeq 0.064$, that translates into $P_{z'}(\theta = 0) = 0.032$.
- It is independent of the hadronic structure of the resonance which cancels in the ratio.

All these properties are modified when we include non-resonant diagrams, but the new contributions correct this result at the level of a few per cent only. In [6] it was shown that the interference of resonance and background leads to an enhancement of the polarization in the vicinity of the resonance. Approximately at four widths below the resonance one gets a polarization five times bigger, but the number of events decreases by three orders of magnitude. It is more efficient from statistics to stay on the peak of the resonance.

Table 1. Dominant $\Upsilon(1S)$ decay channels

$\Upsilon(1S)$ $\quad I^G(J^{PC}) = 0^-(1^{--})$	
Mass $M_\Upsilon = 9460.37 \pm 0.21\ MeV$	
Width $\Gamma = 52.5 \pm 1.8\ KeV$	
Decay modes	Fraction Γ_i/Γ
$\tau^+\tau^-$	$(2.67^{+0.14}_{-0.16})\%$
$\mu^+\mu^-$	$(2.52 \pm 0.17)\%$
e^+e^-	$(2.48 \pm 0.07)\%$
$J/\psi(1S)\ anything$	$(1.1 \pm 0.4) \times 10^{-3}$
$\gamma\ 2h^+2h^-$	$(7.0 \pm 1.5) \times 10^{-4}$
$\gamma\ 3h^+3h^-$	$(5.4 \pm 2.0) \times 10^{-4}$
$\gamma\ 4h^+4h^-$	$(7.4 \pm 3.5) \times 10^{-4}$

The above example indicates that a more detailed analysis of the problem is worth. In the next section we calculate the complete $\tau^-\tau^+$ density matrix in this process which contains all the relevant information on g_V^b. In Sect. 3 we study the hadronic decays of the τ to analyze the τ density matrix. From here, Sect. 4 is devoted to the statistical accuracy that is possible to reach in the measurement of the vector coupling to the b quark in each channel.

2 $\tau^+\tau^-$ density matrix

The density matrix from $e^-(\xi_-, \mathbf{l}_-)e^+(\xi_+, \mathbf{l}_+) \to \tau^-(\lambda_-, \mathbf{p}_-)\tau^+(\lambda_+, \mathbf{p}_+)$, in terms of helicity amplitudes [7], when the initial beams are unpolarized, is given by

$$\rho^\tau_{(\lambda_-,\lambda_+),(\lambda'_-,\lambda'_+)} = \sum_\xi f_{(\lambda_-,\lambda_+),\xi}\ (\theta) f^*_{(\lambda'_-,\lambda'_+),\xi}\ (\theta) \quad (2)$$

where the angle (θ) is given by the direction of the τ^- relative to the initial e^- beam, with the x-z plane defined as the scattering plane.

Using reduced helicity amplitudes, $T^J_{\lambda\xi}$, and rotation matrices these helicity amplitudes are [7],

$$f_{\lambda,\xi}\ (\theta) = d^J_{\xi,\lambda}(\theta)\, T^J_{\lambda\xi} \quad (3)$$

where $\lambda = \lambda_- - \lambda_+$, $\xi = \xi_- - \xi_+$ and $d^J_{\xi,\lambda}(\theta)$ are the reduced rotation matrices around the y-axis. If we neglect the electron mass, the total angular momentum of the process is always $J = 1$. Therefore we get (3) where we have re-defined our reduced helicity amplitudes with respect to [7], including in our definition several normalization factors irrelevant in our analysis [4]. Furthermore, helicity conservation in the electron vertex implies that the value of ξ fixes $\xi = (\xi_+, \xi_-)$ too. The reduced helicity amplitudes get contributions from diagrams (1) to (5) in Fig. 1. For the dominant amplitudes which contribute, through interferences, to P-odd and C-odd observables we can write the following terms

$$\begin{aligned} T_{\lambda,\xi} &= T_{\lambda,\xi}(\gamma) + T_{\lambda,\xi}(\gamma Z_A) \\ &\quad + T_{\lambda,\xi}(Z_A\gamma) + T_{\lambda,\xi}(Z_{A,A}) \end{aligned} \quad (4)$$

where $T_{\lambda,\xi}(\gamma)$ accounts for the contribution of diagrams (1) and (2). $T_{\lambda,\xi}(\gamma Z_A)$ is the P-violating piece of diagram (3) plus the VA piece of diagram (5) and $T_{\lambda,\xi}(Z\gamma_A)$ the corresponding P-violating piece of diagram (4) plus the AV piece of diagram (5). Finally $T_{\lambda,\xi}(Z_{A,A})$ is the contribution of diagram (5) with axial couplings in the initial and final vertices. Notice that (4) does not include the $T_{\lambda,\xi}(\gamma Z_V)$, $T_{\lambda,\xi}(Z\gamma_V)$ and $T_{\lambda,\xi}(Z_{V,V})$ pieces, because they are sub-dominant with respect to $T_{\lambda,\xi}(\gamma)$, both in the resonant and non-resonant components.

Taking into account the transformation properties under P of the four amplitudes in (4) we get

$$
\begin{aligned}
&T_{\lambda,\xi}(\gamma) = T_{-\lambda,-\xi}(\gamma) = T_{\lambda,-\xi}(\gamma) = T_{-\lambda,\xi}(\gamma) \\
&T_{\lambda,\xi}(\gamma Z_A) = -T_{-\lambda,-\xi}(\gamma Z_A) = T_{\lambda,-\xi}(\gamma Z_A) \\
&\quad = -T_{-\lambda,\xi}(\gamma Z_A) \\
&T_{\lambda,\xi}(Z_A\gamma) = -T_{-\lambda,-\xi}(Z_A\gamma) = -T_{\lambda,-\xi}(Z_A\gamma) \\
&\quad = T_{-\lambda,\xi}(Z_A\gamma) \\
&T_{\lambda,\xi}(Z_{AA}) = T_{-\lambda,-\xi}(Z_{AA}) = -T_{\lambda,-\xi}(Z_{AA}) \\
&\quad = -T_{-\lambda,\xi}(Z_{AA})
\end{aligned} \tag{5}
$$

where the normalization of the reduced helicity amplitudes is such that

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= Tr(\rho^{out}) = 2\sin^2\theta |T_{(+,+),1}(\gamma)|^2 \\
&+(1+\cos^2\theta)|T_{(+,-),1}(\gamma)|^2 + 4\cos\theta \\
&\times Re\{T_{(+,-),1}(\gamma)T^*_{(+,-),1}(Z_{A,A})\}
\end{aligned} \tag{6}
$$

$$
\sigma = \frac{16\pi}{3}(|T_{(+,+),1}(\gamma)|^2 + |T_{(+,-),1}(\gamma)|^2) \tag{7}
$$

Notice that a C-odd forward-backward asymmetry is generated by the axial couplings of the Z to both electron and τ vertices.

We calculate the independent reduced helicity amplitudes (see (5) for symmetries) from the Feynman diagrams 1-5 of Fig. 1 following the method explained in Appendix B:

$$
\begin{aligned}
&KT_{(+,+),1}(\gamma) = -i4\sqrt{2}\ \frac{e^2}{s}(1+\frac{e^2}{s}Q_b^2|F_\Upsilon|^2 P_\Upsilon)\ M_\tau\ \frac{\sqrt{s}}{2} \\
&KT_{(+,-),1}(\gamma) = -i8\ \frac{e^2}{s}\ (1+\frac{e^2}{s}Q_b^2|F_\Upsilon|^2 P_\Upsilon)\ p^0\ \frac{\sqrt{s}}{2} \\
&KT_{(+,+),1}(\gamma Z_A) = 0 \\
&KT_{(+,-),1}(\gamma Z_A) = -i8\frac{8G_F}{\sqrt{2}}\ g_A \\
&\quad\times(\frac{e^2}{s}Q_b g_V^b|F_\Upsilon|^2 P_\Upsilon - g_V)\ |\mathbf{p}|\ \frac{\sqrt{s}}{2} \\
&KT_{(+,+),1}(Z_A\gamma) = -i4\sqrt{2}\ \frac{8G_F}{\sqrt{2}}\ g_A \\
&\quad\times(\frac{e^2}{s}Q_b g_V^b|F_\Upsilon|^2 P_\Upsilon - g_V)\ M_\tau\ \frac{\sqrt{s}}{2} \\
&KT_{(+,-),1}(Z_A\gamma) = -i8\frac{8G_F}{\sqrt{2}}\ g_A \\
&\quad\times(\frac{e^2}{s}Q_b g_V^b|F_\Upsilon|^2 P_\Upsilon - g_V)\ p^0\ \frac{\sqrt{s}}{2} \\
&KT_{(+,+),1}(Z_{AA}) = 0 \\
&KT_{(+,-),1}(Z_{AA}) = i8\frac{8G_F}{\sqrt{2}}\ g_A^2\ |\mathbf{p}|\ \frac{\sqrt{s}}{2}
\end{aligned} \tag{8}
$$

where K is a constant which takes care of the different normalization of the helicity amplitudes and the Feynman amplitudes. $Q_b = -\frac{1}{3}$ is the charge of the b quark, $g_{V(A)}$ is the vector (axial) coupling to the leptons, P_Υ stands for the Breit-Wigner propagator of the Υ,

$$
P_\Upsilon(s) = \frac{1}{(s-M_\Upsilon^2) + iM_\Upsilon\Gamma_\Upsilon} \tag{9}
$$

and $F_\Upsilon(q^2)$ is the vector form factor defined as,

$$
\begin{aligned}
&\langle\Upsilon(\omega,\mathbf{q})|\bar\psi_b(0)\gamma_\mu\psi_b(0)|0\rangle = \\
&\qquad = F_\Upsilon(q^2)\varepsilon^*_\mu(\omega,\mathbf{q})
\end{aligned} \tag{10}
$$

with $\varepsilon^*_\mu(\omega,\mathbf{q})$, the polarization four-vector. This form factor can be related to the partial width of Υ to e^+e^-,

$$
\Gamma_e = \frac{1}{6\pi}Q_b^2\frac{(4\pi\alpha)^2}{M_\Upsilon^4}|F_\Upsilon|^2\frac{M_\Upsilon}{2} \tag{11}
$$

Notice that all the hadronic uncertainties in our process will be included in this unique form factor. In (8) we can see that the coupling g_V^b appears linearly in the $T_{\lambda,\xi}(\gamma Z_A)$ and $T_{\lambda,\xi}(Z_A\gamma)$ amplitudes. As these two amplitudes contribute, to dominant order, to the observables through interferences with $T_{\lambda,\xi}(\gamma)$, only the P-odd observables contain information on the g_V^b coupling linearly. As a consequence, we analyze the polarizations and the P-odd correlations.

The polarizations of τ^- are given as follows

$$
\begin{aligned}
\frac{d\sigma}{d\Omega}P^{(-)}_{z'}(\theta) &= \rho_{(+,+),(+,+)} + \rho_{(+,-),(+,-)} \\
&\quad -\rho_{(-,+),(-,+)} - \rho_{(-,-),(-,-)} \\
&= 2Re\{T_{(+,-),1}(\gamma)T^*_{(+,-),1}(\gamma Z_A)\} \\
&\quad\times(1+\cos^2\theta) + 4Re\{T_{(+,-),1}(\gamma) \\
&\quad\times T^*_{(+,-),1}(Z_A\gamma)\}\cos\theta
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\frac{d\sigma}{d\Omega}P^{(-)}_{x'}(\theta) &= \rho_{(+,+),(-,+)} + \rho_{(+,-),(-,-)} \\
&\quad +\rho_{(-,+),(+,+)} + \rho_{(-,-),(+,-)} \\
&= -2\sqrt{2}[Re\{T_{(+,+),1}(\gamma) \\
&\quad\times T^*_{(+,-),1}(\gamma Z_A)\}\sin\theta\ \cos\theta \\
&\quad+(Re\{T_{(+,+),1}(\gamma)T^*_{(+,-),1}(Z_A\gamma)\} \\
&\quad+Re\{T_{(+,-),1}(\gamma) \\
&\quad\times T^*_{(+,+),1}(Z_A\gamma)\})\sin\theta]
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\frac{d\sigma}{d\Omega}P^{(-)}_{y'}(\theta) &= -i(\rho_{(+,+),(-,+)} + \rho_{(+,-),(-,-)} \\
&\quad -\rho_{(-,+),(+,+)} - \rho_{(-,-),(+,-)}) \\
&= 2\sqrt{2}[Im\{T_{(+,+),1}(\gamma)T^*_{(+,-),1}(Z_{A,A})\}\sin\theta \\
&\quad+2Im\{T_{(+,+),1}(\gamma) \\
&\quad\times T^*_{(+,-),1}(\gamma)\}\sin\theta\cos\theta]
\end{aligned} \tag{14}
$$

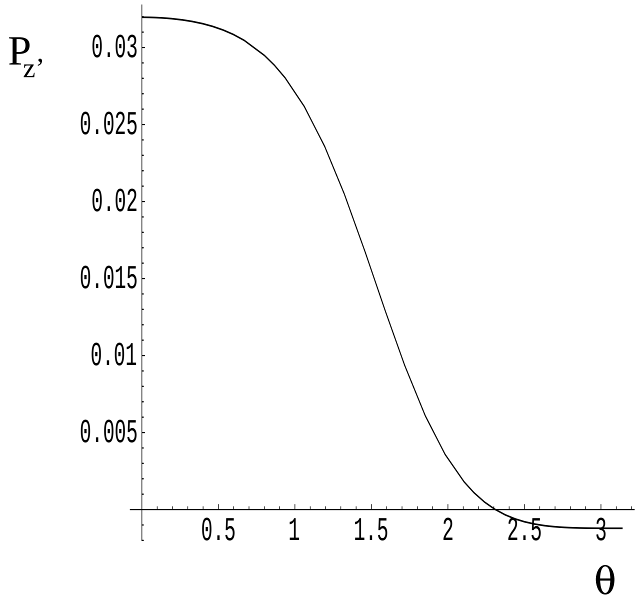

Fig. 2. Longitudinal τ^- polarization, $P_{z'}(\theta)$

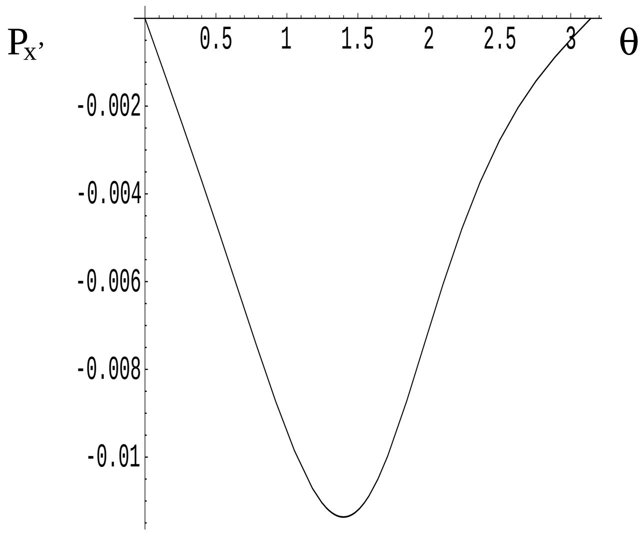

Fig. 3. Transverse τ^- polarization, $P_{x'}(\theta)$

As we can see in these expressions, only $P_{z'}$ and $P_{x'}$ contain information on g_V^b, because both are P-odd, T-even observables. On the contrary, $P_{y'}$ has no information on g_V^b, because it is P-even, T-odd. The T-odd observable $P_{y'}$ needs the interference between resonant and non-resonant amplitudes, which on the Υ peak are relatively imaginary. By the same argument, there is no interference between resonant and non-resonant pieces in the T-even observables, like $P_{z'}$ and $P_{x'}$, on the Υ peak. If we compare the longitudinal polarization, (12), with the result obtained in (1) the difference is in the non-resonant terms proportional to g_V in (8). Thus the new terms are suppressed on the Υ peak by a factor $g_V\alpha^2 Q_b/(9g_V^b b.r.(\Upsilon\to e^+e^-)^2) \approx 3.5\times 10^{-4}$, so that our estimate in (1) is very good. The value of the longitudinal polarization of the τ in the forward direction is $P_{z'}(\theta=0)\simeq .185\cdot g_V^b \simeq 0.032$. In Fig. 2, we plot this polarization as a function of the scattering angle θ.

On the other hand $P_{x'}$ contains similar reduced helicity amplitudes as $P_{z'}$, the only difference is that transverse polarizations are suppressed by a mass insertion, that is a factor $M_\tau/p^0 \simeq .38$. Basically, apart from a different angular dependence, this is the reason that makes this observable less sensitive to g_V^b as we can see in Fig. 3. We get for instance $P_{x'}(\theta=\pi/2)\simeq -.063\cdot g_V^b$, slightly lower than $P_{z'}$.

The information contained in the τ^+ polarizations is closely related to that of the τ^-

$$P_{z'}^{(+)} = -P_{z'}^{(-)};\quad P_{x'}^{(+)} = P_{x'}^{(-)};\quad P_{y'}^{(+)} = -P_{y'}^{(-)} \quad (15)$$

Finally, we also consider the spin correlations between τ^+ and τ^-, defined as follows,

$$\frac{d\sigma}{d\Omega}\mathcal{C}_{ij}(\theta) = Tr(\sigma_i^{(-)}\sigma_j^{(+)}\rho^\tau)$$

With this definition it is easy to see that the only P-odd observables are the $\mathcal{C}_{x,y}$, $\mathcal{C}_{z,y}$, $\mathcal{C}_{y,x}$ and $\mathcal{C}_{y,z}$ correlations,

$$\begin{aligned}\frac{d\sigma}{d\Omega}\mathcal{C}_{zy}(\theta) &= i(-\rho_{(+,+),(+,-)} \\ &\quad +\rho_{(-,+),(-,-)} + \rho_{(+,-),(+,+)} - \rho_{(-,-),(-,+)}) \\ &= -2\sqrt{2}[Im\{T_{(+,+),1}(\gamma) \\ &\quad \times T^*_{(+,-),1}(\gamma Z_A)\}\sin\theta\ \cos\theta \\ &\quad +(Im\{T_{(+,+),1}(\gamma)T^*_{(+,-),1}(Z_A\gamma)\} \\ &\quad -Im\{T_{(+,-),1}(\gamma)T^*_{(+,+),1}(Z_A\gamma)\})\sin\theta] \quad (16)\end{aligned}$$

$$\mathcal{C}_{yz}(\theta) = \mathcal{C}_{zy}(\theta) \quad (17)$$

$$\begin{aligned}\frac{d\sigma}{d\Omega}\mathcal{C}_{xy}(\theta) &= i(-\rho_{(+,+),(-,-)} - \rho_{(-,+),(+,-)} \\ &\quad +\rho_{(+,-),(-,+)} + \rho_{(-,-),(+,+)}) \quad (18) \\ &= 2Im\{T_{(+,-),1}(\gamma)T^*_{(+,-),1}(\gamma Z_A)\}\sin^2\theta\end{aligned}$$

$$\mathcal{C}_{yx}(\theta) = -\mathcal{C}_{xy}(\theta)$$

As we pointed out before all the relevant observables are associated to the amplitudes $T_{\lambda,\xi}(\gamma Z_A)$ and $T_{\lambda,\xi}(Z_A\gamma)$, for which their strength relative to the dominant term is set by the factor $8G_F M_\Upsilon^2/(4\pi\alpha\sqrt{2}) \simeq 0.064$. Unfortunately, the P-odd correlations are also T-odd so that they need an imaginary part. With our set of reduced helicity amplitudes, it becomes necessary to have an interference between resonant and non-resonant diagrams to get an imaginary part. Therefore, these contributions have the same suppression that diagram (1) with respect to diagram (2), that is, $\alpha/(3\ b.r.(\Upsilon\to e^+e^-)) \simeq 1/10$. Notice also that the helicity structure of $\mathcal{C}_{zy}$ is very similar to $P_{x'}$, and so it has the same suppression factor, unlike $\mathcal{C}_{xy}$ which is not helicity suppressed.

Table 2. Dominant τ decay channels

$\tau \quad J = \frac{1}{2}$	
Mass $M_\tau = 1777.00^{+0.30}_{-0.27}\ MeV$	
Mean life $\tau = (291.0 \pm 1.5) \times 10^{-15}\ s$	
Decay modes	Fraction Γ_i/Γ
$\mu^- \bar{\nu}_\mu \nu_\tau$	$(17.35 \pm 0.10)\%$
$e^- \bar{\nu}_e \nu_\tau$	$(17.83 \pm 0.08)\%$
$\pi^- \nu_\tau$	$(11.31 \pm 0.15)\%$
$\pi^- \pi^0 \nu_\tau$	$(25.24 \pm 0.16)\%$
$h^- 2\pi^0 \nu_\tau$	$(9.50 \pm 0.14)\%$
$h^- h^- h^+ \nu_\tau$	$(9.80 \pm 0.10)\%$

To conclude: the main P-odd observables able to separate g_V^b are the longitudinal polarizations, $P_{z'}$, of both τ's, then the transverse polarization $P_{x'}$ and finally $\mathcal{C}_{xy}$ and $\mathcal{C}_{zy}$ ordered from the most relevant to the least one.

In the next section we connect the observables of this section to measurable quantities, analyzing the angular distributions of the decay products of the two τ.

3 Decay of a polarized τ

The main τ decay channels are presented in Table 2. The purely leptonic decays have branching ratios of 17.35% for muons and 17.83% for electrons. Unfortunately, these decay modes have two neutrinos in the final state, which implies that the τ direction can not be reconstructed. Then, their sensitivity to the τ polarization is small [8] compared with the hadronic decays.

So we concentrate on the hadronic τ decays which have only one neutrino in the final state and allow to reconstruct the τ direction if both τ´s decay hadronically [9]. These decays are $\tau^- \to \pi^- \nu_\tau$, with a branching ratio of 11.31%, $\tau^- \to \rho^- \nu_\tau$ which corresponds almost exactly to the two pions channel, with branching ratio 25.24%, and $\tau^- \to a_1^- \nu_\tau$ which is given by the sum of the three pion final states. In this work we analyze the decays $\tau^- \to \pi^- \nu_\tau$ and $\tau^- \to \rho^- \nu_\tau$. Other τ decay channels have been studied elsewhere, for instance $\tau \to a_1 \nu_\tau$ can be found in [10] for LEP physics,

3.1 $\tau^- \to \pi^- \nu_\tau$

This channel has been used for a long time to measure the τ polarization because of its good sensitivity. The differential decay width is given by,

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\Omega} = \frac{1}{4\pi}[1 + \mathbf{P}\hat{k}(\Omega)] \tag{19}$$

where $\hat{k}$ is the unit vector in the direction of the pion.

Equation (19) shows the good sensitivity to the polarization from the angular distribution of the decay pions. This is due to the fact that in this process there is only one reduced helicity amplitude. Then, the angular distribution is necessarily proportional to this unique helicity amplitude which can be factorized. This is the reason for the popularity of this channel as polarization analyzer in τ decays.

With these elements, following Appendix A, we build a complete two steps angular distribution [4,11] for the whole process, $e^- e^+ \to \tau^- \tau^+ \to (\pi^- \nu_\tau)(\pi^+ \bar{\nu}_\tau)$

$$\begin{aligned}\frac{d\sigma}{d\Omega\ d\Omega_+\ d\Omega_-} &= \frac{d\sigma}{d\Omega}\ (1\ +\ \mathbf{P}^{(-)} \cdot \hat{k}_-\ -\ \mathbf{P}^{(+)} \cdot \hat{k}_+ \\ &-\mathcal{C}_{zz} \cos\theta_- \cos\theta_+\ -\ \mathcal{C}_{xx} \sin\theta_- \cos\phi_- \sin\theta_+ \cos\phi_+ \\ &-\mathcal{C}_{yy} \sin\theta_- \sin\phi_- \sin\theta_+ \sin\phi_+ \\ &-\mathcal{C}_{zx} \cos\theta_- \sin\theta_+ \cos\phi_+ - \mathcal{C}_{xz} \sin\theta_- \cos\phi_- \cos\theta_+ \\ &-\mathcal{C}_{zy} \cos\theta_- \sin\theta_+ \sin\phi_+ - \mathcal{C}_{yz} \sin\theta_- \sin\phi_- \cos\theta_+ \\ &-\ \mathcal{C}_{xy} \sin\theta_- \cos\phi_- \sin\theta_+ \sin\phi_+ \\ &-\mathcal{C}_{yx} \sin\theta_- \sin\phi_- \sin\theta_+ \cos\phi_+)\end{aligned} \tag{20}$$

where $(\theta_\pm, \phi_\pm)$ is the direction of the $\pi^\pm$ in the rest frame of the $\tau^\pm$ and $\hat{k}_\pm$ is the unit vector in this direction.

This is the cross section we have to study to extract g_V^b from this channel. In Sect. 4 we analyze the sensitivity of the different observables constructed in Sect. 2 from this cross section.

3.2 $\tau^- \to \rho^- \nu_\tau \to \pi^- \pi^0 \nu_\tau$

In the channel $\tau \to \rho \nu_\tau$ we have a spin 1 particle in the final state. This implies that we have two different helicity amplitudes for the decay, and so different combinations of these two amplitudes enter in the polarized and unpolarized pieces,

$$\frac{1}{\Gamma_{\tau\to\rho\nu}}\frac{d\Gamma_{\tau\to\rho\nu}}{d\Omega} = \frac{1}{4\pi}[1 + \alpha_\rho \mathbf{P}\hat{k}(\Omega)] \tag{21}$$

where α_ρ is a ratio of reduced helicity amplitudes, T_ν^j defined below in (27,28,29), which we get in terms of the masses as,

$$\begin{aligned}\alpha_\rho &= \frac{|T_{0,-1/2}|^2 - |T_{-1,-1/2}|^2}{|T_{0,-1/2}|^2 + |T_{-1,-1/2}|^2} \\ &= \frac{M_\tau^2 - 2M_\rho^2}{M_\tau^2 + 2M_\rho^2} = 0.456\end{aligned} \tag{22}$$

Then, in spite of its bigger branching ratio, the sensitivity of the ρ channel at this level is smaller than in $\tau \to \pi \nu_\tau$. However, as pointed out in [8], this situation can be improved if, in addition, we try to get some extra information on the ρ helicity. To do this, we include another step in this chain of decays, and analyze the decay $\rho^- \to \pi^- \pi^0$. The cross section for the whole process, $e^- e^+ \to \tau^- \tau^+ \to ((\pi^- \pi^0)\nu_\tau)(\rho^+ \bar{\nu}_\tau)$, can be written as,

$$\begin{aligned}\frac{d\sigma}{d\Omega d\Omega_1^- d\Omega_1^+ d\Omega_2} &= \frac{d\sigma}{d\Omega}(e^+ e^- \to \tau^+ \tau^-) \\ &\times \frac{1}{\Gamma_\tau^2}\frac{d\Gamma_{\tau^-\tau^+\to\rho^-\nu\rho^+\bar{\nu}}}{d\Omega_1^- d\Omega_1^+} \\ &\times \frac{1}{\Gamma_\rho}\frac{d\Gamma_{\rho^-\to\pi^-\pi^0}}{d\Omega_2}\end{aligned} \tag{23}$$

where the $\tau^-\tau^+$ decay amplitude to $\rho^-\nu\rho^+\bar{\nu}$ is given below in (33), and we have added the last factor which is the decay width of a polarized and aligned ρ into two pions. The expression for this decay width is

$$\frac{1}{\Gamma_{\rho\to\pi\pi}} \frac{d\Gamma_{\rho\to\pi\pi}}{d\Omega_2} = \frac{1}{4\pi} \times[1-\sqrt{10}\sum_N \mathcal{D}^{(2)}_{N,0}(\phi_2,\theta_2,0)t_{2,N}] \quad (24)$$

where $t_{2,N}$ are the tensor polarizations of the ρ.

In Eq(24), we see that the ρ polarizations do not appear in the decay angular distribution because $\rho\to\pi\pi$ is a strong decay and therefore P-conserving. The alignments, higher order multipole parameters, do appear but the polarizations do not.

Now, we calculate the density matrix for a ρ coming from the decay of a polarized τ in its center of mass frame, and then apply the necessary Wigner rotation [12], as explained in App. A, to get the density matrix in a frame where the τ is boosted.

The density matrix of a single ρ from the decay of one of the two τs also contains information on the decay of the other τ if we study the correlations and do not integrate the second τ decay. So, as we are interested in the measurement of correlations between the two τ´s, we study the ρ^- density matrix in a decay $\tau^-\tau^+\to(\rho^-\nu_\tau)(\rho^+\bar{\nu}_\tau)$ and in a decay $\tau^-\tau^+\to(\rho^-\nu_\tau)(\pi^+\bar{\nu}_\tau)$. Naturally these two different density matrices will coincide when we integrate completely the τ^+ decay products.

Following Appendix A, we can write the complete ρ^- density matrix from a decay $\tau^-\tau^+\to(\rho^-\nu_\tau)(\rho^+\bar{\nu}_\tau)$ as

$$\begin{aligned}\rho_{\mu_-\mu'_-} = &\sum_{\lambda_+\lambda_-\lambda'_+\lambda'_-}\sum_{\nu_-\nu'_-\nu_+} f^{(+)}_{(\nu_+1/2)\lambda_+}(\Omega_+)d_{\mu_-\nu_-}(\omega_-)f^{(-)}_{(\nu_--1/2)\lambda_-}(\Omega_-)\\ &\times\rho^{\tau}_{(\lambda_-,\lambda_+),(\lambda'_-,\lambda'_+)}f^{(+)*}_{(\nu_+1/2)\lambda'_+}(\Omega_+)\\ &\times d_{\mu'_-\nu'_-}(\omega_-)f^{(-)*}_{(\nu'_--1/2)\lambda'_-}(\Omega_-)\end{aligned} \quad (25)$$

where we have used a reference system in LAB frame with the τ^- in the z-axis and the initial beams in the x-z plane, to simplify the expressions for the Wigner rotations.

We define an effective density matrix $\bar{\rho}$ without the Wigner rotations that, if integrated over the Ω_+ variables, would correspond to the density matrix in the τ^- rest frame, which is

$$\rho_{\mu_-\mu'_-} = d_{\mu_-\nu_-}(\omega_-)\bar{\rho}_{\nu_-\nu'_-}d_{\mu'_-\nu'_-}(\omega_-) \quad (26)$$

The next step is to calculate this effective density matrix in terms of reduced helicity amplitudes. In a decay we define the helicity amplitudes as,

$$f_{\nu\lambda}(\theta,\phi) = \sqrt{\frac{2j+1}{4\pi}}\mathcal{D}^{j*}_{\lambda,\nu_1-\nu_2}(\phi,\theta,0)T^j_\nu \quad (27)$$

With this definition and (25) and (26), we get our effective density matrix. In this process, $\tau^-(\lambda_-,\mathbf{0})\to\rho^-(\sigma,\mathbf{k}_-)\nu_\tau(-1/2,-\mathbf{k}_-)$, we have only two reduced helicity amplitudes T_ν if we take $M_\nu=0$

$$\begin{aligned}KT_{-1,-1/2} &= i4G_F\ V_{ud}\ F_\rho(k_-^2)\ \sqrt{M_\tau k_-}\\ KT_{0,-1/2} &= i2\sqrt{2}G_F\ V_{ud}\ F_\rho(k_-^2)\ \sqrt{M_\tau k_-}\ \frac{M_\tau}{M_\rho}\end{aligned} \quad (28)$$

Similarly for the decay $\tau^+(\lambda_+,\mathbf{0})\to\rho^+(\sigma,\mathbf{k}_+)\bar{\nu}_\tau(1/2,-\mathbf{k}_+)$,

$$\begin{aligned}KT_{1,1/2} &= -i4G_F\ V_{ud}\ F_\rho(k_+^2)\ \sqrt{M_\tau k_+}\\ KT_{0,1/2} &= i2\sqrt{2}G_F\ V_{ud}\ F_\rho(k_+^2)\ \sqrt{M_\tau k_+}\ \frac{M_\tau}{M_\rho}\end{aligned} \quad (29)$$

where $F_\rho(k_\pm^2)$ is a form factor defined as,

$$\langle\rho^0(\sigma,\mathbf{k})|\frac{1}{2}(\bar{\psi}_u(0)\gamma_\mu\psi_u(0)-\bar{\psi}_d(0)\gamma_\mu\psi_d(0))|0\rangle = F_\rho(k^2)\varepsilon^*_\mu(\sigma,\mathbf{k}) \quad (30)$$

and this form factor can be related with the form factors for the charged ρ‘s through the Wigner-Eckart theorem,

$$\langle\rho^\pm(\sigma,\mathbf{k})|\frac{1}{\sqrt{2}}\bar{\psi}_{u,d}(0)\gamma_\mu\psi_{d,u}(0)|0\rangle = F_\pm(k^2)\varepsilon^*_\mu(\sigma,\mathbf{k}) \quad (31)$$

$$F_\rho(k^2) = -F_+(k^2) = -F_-(k^2)$$

and $\varepsilon_\mu(\sigma,k)$ is the polarization four-vector.

The total rate for $\tau^-\to\rho^-\nu_\tau$ is,

$$\Gamma_{\tau\to\rho\upsilon} = \frac{1}{2M_\tau}(|T_{-1,-1/2}|^2+|T_{0,-1/2}|^2) \quad (32)$$

Following Appendix A, we calculate the C.M. multipole parameters corresponding to the density matrix $\bar{\rho}$ in terms of the reduced amplitudes of (28,29), taking into account that we do not integrate the direction of the second ρ. The angular distribution is

$$\begin{aligned}\frac{d\Gamma_{\tau^-\tau^+}}{d\Omega_1^-d\Omega_1^+} &= Tr\{\rho\} = \rho_{-1,-1}+\rho_{0,0}\\ &= (|T_{-1,-1/2}|^2+|T_{0,-1/2}|^2)(|T_{1,1/2}|^2+|T_{0,1/2}|^2)\\ &[1+\alpha_\rho\bar{P}\cdot\hat{k}^- - \bar{\alpha}\bar{P}\cdot\hat{k}^+ - \alpha_\rho\bar{\alpha}(\mathcal{C}_{zz}\cos\theta_-\cos\theta_+\\ &+\mathcal{C}_{xx}\sin\theta_-\cos\phi_-\sin\theta_+\cos\phi_+\\ &+\mathcal{C}_{yy}\sin\theta_-\sin\phi_-\sin\theta_+\sin\phi_+\\ &+\mathcal{C}_{zx}\cos\theta_-\sin\theta_+\cos\phi_+ +\mathcal{C}_{xz}\sin\theta_-\cos\phi_-\cos\theta_+\\ &+\mathcal{C}_{zy}\cos\theta_-\sin\theta_+\sin\phi_+ +\mathcal{C}_{yz}\sin\theta_-\sin\phi_-\cos\theta_+\\ &+\mathcal{C}_{xy}\sin\theta_-\cos\phi_-\sin\theta_+\sin\phi_+\\ &+\mathcal{C}_{yx}\sin\theta_-\sin\phi_-\sin\theta_+\cos\phi_+)]\end{aligned} \quad (33)$$

Here $(\theta_\pm,\phi_\pm)$ is the direction of the $\rho^\pm$ in the rest frame of the $\tau^\pm$ and $\hat{k}_\pm$ is the unit vector in this direction. α_ρ was defined in (22) and $\bar{\alpha}$ is equal to α_ρ if the τ^+ decays to $\rho^+\bar{\nu}_\tau$, but equal to 1 if it decays to $\pi^+\bar{\nu}_\tau$. This notation holds for all observables that follow. The tensor polarization $\bar{t}_{20}$ is

$$Tr\{\rho\}\bar{t}_{20} = \sqrt{\frac{1}{10}}(\rho_{-1,-1} - 2\rho_{0,0})$$
$$= \sqrt{\frac{1}{10}}(|T_{-1,-1/2}|^2 + |T_{0,-1/2}|^2)$$
$$\times(|T_{1,1/2}|^2 + |T_{0,1/2}|^2)[\gamma_\rho - \beta_\rho \bar{P}^- \cdot \hat{k}^- - \bar{\alpha}\gamma_\rho \bar{P}^+ \cdot \hat{k}^+$$
$$+\bar{\alpha}\beta_\rho(\mathcal{C}_{zz}\cos\theta_-\cos\theta_+$$
$$+\mathcal{C}_{xx}\sin\theta_-\cos\phi_-\sin\theta_+\cos\phi_+$$
$$+\mathcal{C}_{yy}\sin\theta_-\sin\phi_-\sin\theta_+\sin\phi_+$$
$$+\mathcal{C}_{zx}\cos\theta_-\sin\theta_+\cos\phi_+$$
$$+\mathcal{C}_{xz}\sin\theta_-\cos\phi_-\cos\theta_+ + \mathcal{C}_{zy}\cos\theta_-\sin\theta_+\sin\phi_+$$
$$+\mathcal{C}_{yz}\sin\theta_-\sin\phi_-\cos\theta_+$$
$$+\mathcal{C}_{xy}\sin\theta_-\cos\phi_-\sin\theta_+\sin\phi_+$$
$$+\mathcal{C}_{yx}\sin\theta_-\sin\phi_-\sin\theta_+\cos\phi_+)] \tag{34}$$

where we have introduced two new coefficients β_ρ and γ_ρ defined as

$$\gamma_\rho = \frac{|T_{-1,-1/2}|^2 - 2|T_{0,-1/2}|^2}{|T_{0,-1/2}|^2 + |T_{-1,-1/2}|^2} = \frac{2M_\rho^2 - 2M_\tau^2}{M_\tau^2 + 2M_\rho^2} = -1.18 \tag{35}$$

$$\beta_\rho = \frac{|T_{-1,-1/2}|^2 + 2|T_{0,-1/2}|^2}{|T_{0,-1/2}|^2 + |T_{-1,-1/2}|^2} = \frac{2M_\rho^2 + 2M_\tau^2}{M_\tau^2 + 2M_\rho^2} = 1.73 \tag{36}$$

The other tensor polarizations $\bar{t}_{21}$ and $\bar{t}_{22}$ are,

$$Tr\{\rho\}\bar{t}_{21} = \sqrt{\frac{3}{10}}\rho_{-1,0} = \sqrt{\frac{3}{10}}(|T_{-1,-1/2}|^2 + |T_{0,-1/2}|^2)$$
$$\times(|T_{1,1/2}|^2 + |T_{0,1/2}|^2)[\delta_\rho(P_z^-(-\sin\theta_-)$$
$$+P_x^-(\cos\theta_-\cos\phi_- - i\sin\phi_-)$$
$$+P_y^-(\cos\theta_-\sin\phi_- + i\cos\phi_-))$$
$$-\delta_\rho\bar{\alpha}(\mathcal{C}_{zz}(-\sin\theta_-)\cos\theta_+$$
$$+\mathcal{C}_{xx}(\cos\theta_-\cos\phi_- - i\sin\phi_-)\sin\theta_+\cos\phi_+$$
$$+\mathcal{C}_{yy}(\cos\theta_-\sin\phi_- + i\cos\phi_-)\sin\theta_+\sin\phi_+$$
$$+\mathcal{C}_{zx}(-\sin\theta_-)\sin\theta_+\cos\phi_+$$
$$+\mathcal{C}_{xz}(\cos\theta_-\cos\phi_- - i\sin\phi_-)\cos\theta_+$$
$$+\mathcal{C}_{zy}(-\sin\theta_-)\sin\theta_+\sin\phi_+$$
$$+\mathcal{C}_{yz}(\cos\theta_-\sin\phi_- + i\cos\phi_-)\cos\theta_+$$
$$+\mathcal{C}_{xy}(\cos\theta_-\cos\phi_- - i\sin\phi_-)\sin\theta_+\sin\phi_+$$
$$+\mathcal{C}_{yx}(\cos\theta_-\sin\phi_- + i\cos\phi_-)\sin\theta_+\cos\phi_+)] \tag{37}$$

$$Tr\{\rho\}\bar{t}_{22} = \sqrt{\frac{3}{5}}\rho_{-1,1} = 0 \tag{38}$$

where we have introduced a new coefficient

$$\delta_\rho = \frac{T_{-1,-1/2}T^*_{0,-1/2}}{|T_{0,-1/2}|^2 + |T_{-1,-1/2}|^2} = \frac{\sqrt{2}M_\rho M_\tau}{M_\tau^2 + 2M_\rho^2} = 0.445 \tag{39}$$

In general, these multipole parameters can be complex, as $\bar{t}_{21}$.

It is very important to notice that unlike α_ρ and δ_ρ, that are small, both γ_ρ and β_ρ are bigger than one in magnitude and so they can enhance the information on the τ polarizations.

As we have already pointed out, the only difference between the decay distributions for $\tau^-\tau^+ \to (\rho^-\nu_\tau)(\rho^+\bar{\nu}_\tau)$ and $\tau^-\tau^+ \to (\rho^-\nu_\tau)(\pi^+\bar{\nu}_\tau)$ is the value of the coefficient $\bar{\alpha}$

$$\tau^+ \to \rho^+\bar{\nu}_\tau \Longrightarrow \bar{\alpha} = .456$$
$$\tau^+ \to \pi^+\bar{\nu}_\tau \Longrightarrow \bar{\alpha} = 1 \tag{40}$$

The final step to get the alignments appearing in (24) and (23) needs to perform the Wigner rotation, (13).

$$t_{20} = d^2_{0M}(\omega_-)\bar{t}_{2M} = \bar{t}_{20}(\frac{3}{2}\cos^2\omega_- - \frac{1}{2}) + \sqrt{6}Re\{\bar{t}_{21}\}\sin\omega_-\cos\omega_- \tag{41}$$

$$t_{21} = d^2_{1M}(\omega_-)\bar{t}_{2M} = -\bar{t}_{20}\cos\omega_-\sin\omega_- + iIm\{\bar{t}_{21}\}\cos\omega_- + Re\{\bar{t}_{21}\}(\cos^2\omega_- - \sin^2\omega_-) \tag{42}$$

$$t_{22} = d^2_{2M}(\omega_-)\bar{t}_{2M} = \bar{t}_{20}\frac{\sqrt{6}}{4}\sin^2\omega_- - iIm\{\bar{t}_{21}\}\sin\omega_- - Re\{\bar{t}_{21}\}\cos\omega_-\sin\omega_- \tag{43}$$

where ω is the Wigner rotation associated with the boost from the τ rest frame to the e^+e^- C.M. frame which transforms the ρ four-momentum, $k'_\rho = \Lambda k_\rho$. These rotations have the following expression

$$\sin\omega = \frac{M_\rho|\mathbf{p}_\tau|\sin\theta}{M_\tau \mid \mathbf{k}'_\rho \mid} \quad , \quad \cos\omega = \frac{E_\rho E'_\rho M_\tau - E_\tau M_\rho^2}{\mid \mathbf{k}_\rho \mid\mid \mathbf{k}'_\rho \mid M_\tau}$$

with $(E_\tau, \mathbf{p}_\tau)$ the τ four-momentum in the e^+e^- C.M. frame and θ the angle between the ρ and the direction of the boost in the τ rest frame.

We have thus completed the angular distributions of the different chains with final states:

- $(\pi^-\nu_\tau)(\pi^+\bar{\nu}_\tau)$
- $(\pi^-\pi^0\nu_\tau)(\pi^+\bar{\nu}_\tau) + (\pi^+\pi^0\nu_\tau)(\pi^-\bar{\nu}_\tau)$
- $(\pi^-\pi^0\nu_\tau)(\rho^+\bar{\nu}_\tau) + (\pi^+\pi^0\nu_\tau)(\rho^-\bar{\nu}_\tau)$

Next we find the statistical accuracy one can obtain in the measurement of g_V^b

4 Statistical sensitivity

To estimate the obtainable precision in the measurement of g_V^b in a B-meson facility, we use the formalism of references [13–15]. In these references they call "ideal statistical error" of a parameter p which enters a function $f(x, y; p)$ to be determined experimentally, to the error obtained from a least-squares-fit to this function with N events. To obtain this error, we use that for large number of events, N, the likelihood function approaches a gaussian. Then, if the function $f(x, y; p)$ is normalized to one on the physical region, the ideal statistical error is given by

$$\sigma_p^2 = \frac{1}{N}[\int (\frac{\delta \ln f(x_1, \ldots, x_n; p)}{\delta p})^2 \cdot f(x_1, ..., x_n; p) dx_1 \ldots dx_n]^{-1} \quad (44)$$

The word "ideal" stands for the fact that we are not considering the efficiency of the detectors, effects of finite experimental resolution and we assume an ideal distribution of the N events according to $f(x, y; p)$.

In this case our function $f(x, y; p)$ will be the normalized cross section to the different channels. First we study the reaction $e^-e^+ \to \tau^-\tau^+ \to (\pi^-\nu_\tau)(\pi^+\bar{\nu}_\tau)$. Our result is, in this channel,

$$\sigma_{g_V^b} = \frac{11.1}{\sqrt{N}} \quad (45)$$

where N is the number of $e^-e^+ \to \tau^-\tau^+ \to (\pi^-\nu_\tau)(\pi^+\bar{\nu}_\tau)$ events. This means that in a B-meson facility with 10^8 Υ $(1S)$ produced per year one could get a sensitivity to g_V^b of $6 \cdot 10^{-2}$ only with this channel. Equation (20) gives all information available in the process, but not all these observables will be useful for our purposes. In particular the P-even, T-even correlations $(\mathcal{C}_{zz}, \mathcal{C}_{xx}, \mathcal{C}_{yy}, \mathcal{C}_{zx})$ get contributions from $|T_{\lambda,\xi}(\gamma)|^2$, so they are order 1 and they have no information on g_V^b. On the other hand, the P-odd, T-odd correlations are not as sensitive to this parameter as the polarizations. We can ask whether we can improve the sensitivity by integrating out some of the final state variables. From (20) we would like to eliminate the P-even correlations maintaining the polarizations and P-odd correlations, but unfortunately we can not achieve this result integrating out some final state variables. The only interesting possibility is to consider both τ decays independently, integrating out one of the two pion directions with respect to the τ. In this case we are considering the $e^-e^+ \to \tau^-\tau^+ \to ((\pi^-\nu_\tau)\tau^+ + \tau^-(\pi^+\bar{\nu}_\tau))$ events, where both τ are hadronically reconstructed. By doing this we increase considerably the number of events, because now we include also the events in which the second τ decays to ρ and a_1. This is approximately a factor of 5 in the number of events for each tau, but we have two independent events for each Υ decay, then the number of events increases by a factor of 10. The sensitivity one can get with this new decay distribution (we obtain it directly from (20) integrating out (θ_+, ϕ_+)) is

$$\sigma_{g_V^b} = \frac{14.6}{\sqrt{N}} \quad (46)$$

where N is the number of $e^-e^+ \to \tau^-\tau^+ \to ((\pi^-\nu_\tau)\tau^+ + \tau^-(\pi^+\bar{\nu}_\tau))$ events. We can see that the difference between (45) and (46) is roughly a factor $\sqrt{2}$, this is due to the fact that in (45) we included both the τ^+ and τ^- polarizations while in (46) we take into account only one of them. On the other hand, this means that the P-odd correlations do not improve the measurement of g_V^b in a significant way. Again with $10^8 \Upsilon$ per year, one can get a sensitivity of $2.3 \cdot 10^{-2}$. Even more, the simplest polarization analyzer we can use is the energy of the pions. The energy of the pions in LAB is related with the angle in C.M. of the τ

$$E_\pi = \frac{E_\pi^* E_\tau + q k_\pi^* \cos\theta_-}{M_\tau} \quad (47)$$

where $(E_\pi^*, \mathbf{k}_\pi^*(\theta_-))$ is the four-momentum of the pion in the τ C.M., and $(E_\tau, \mathbf{q})$ the four-momentum of the τ in LAB. Again from (20), if we integrate all the angular variables but θ_- and make this change of variable to E_π one gets a sensitivity of

$$\sigma_{g_V^b} = \frac{20.9}{\sqrt{N}} \quad (48)$$

And, in this way, we do not put any restriction on the second τ decay, then with $10^8\Upsilon$ per year one can get a sensitivity to g_V^b of $2.7 \cdot 10^{-2}$, simply using the $\pi\nu$ channel and measuring only the pion energy. From this point of view, it is evident that, if we consider only the $\pi\nu$ decay channel, this is the best strategy to measure g_V^b, because one can use all the $\tau \to \pi\nu$ events and is experimentally simpler.

We have also studied the channel $\tau \to \rho\nu$ as a polarization analyzer. Then if we apply (44) to the complete distribution $e^-e^+ \to \tau^-\tau^+ \to ((\pi^-\pi^0)\nu_\tau)(\rho^+\bar{\nu}_\tau) + ((\pi^+\pi^0)\nu_\tau)(\rho^-\bar{\nu}_\tau)$ given by (23) one gets

$$\sigma_{g_V^b} = \frac{12.4}{\sqrt{N}} \quad (49)$$

that is slightly worse than our result in the $e^-e^+ \to \tau^-\tau^+ \to (\pi^-\nu_\tau)(\pi^+\bar{\nu}_\tau)$ channel, but now we have more events because this channel has a bigger branching ratio. This translates in a B-meson facility with $10^8\Upsilon$ produced per year in a sensitivity to g_V^b of $2.1 \cdot 10^{-2}$. Other possibility is to use a combined channel as $e^-e^+ \to \tau^-\tau^+ \to (((\pi^-\pi^0)\nu_\tau)(\pi^+\bar{\nu}_\tau) + (\pi^-\nu_\tau)((\pi^+\pi^0)\bar{\nu}_\tau))$ where the sensitivity to the polarization of the $\tau^+ \to \pi^+\bar{\nu}_\tau$ is better than $\tau^+ \to \rho^+\bar{\nu}_\tau$ if we do not analyze the next decay $\rho^+ \to \pi^+\pi^0$. The result is,

$$\sigma_{g_V^b} = \frac{9.1}{\sqrt{N}} \quad (50)$$

and the number of events is similar to the previous case. The sensitivity to g_V^b that one can reach in this channel, with $10^8\Upsilon$, is $2.3 \cdot 10^{-2}$. As in the π channel we can increase the statistics by integrating one of the decays, although some sensitivity is lost. We integrate the direction of one of the ρ or equivalently the π direction in (23) and only require this τ to decay hadronically. Then we keep simply the decay of a τ to $\rho\nu$ and then the decay of ρ to $\pi\pi$. Then we get a sensitivity of,

$$\sigma_{g_V^b} = \frac{14.6}{\sqrt{N}} \quad (51)$$

but now the number of events has increased a factor of 10, because N is the number of $e^-e^+ \to \tau^-\tau^+ \to (((\pi^-\pi^0)\nu_\tau)\tau^+ + \tau^-((\pi^+\pi^0)\bar{\nu}_\tau))$ events. Again with $10^8\Upsilon$ per year, one gets a sensitivity of $1.6 \cdot 10^{-2}$.

After this analysis we can combine a series of independent measurements into a final value for g_V^b with an error given by

$$\sigma = (\sum_i \frac{1}{\sigma_i^2})^{-1/2} \tag{52}$$

then we can combine the error obtained with the channels $e^-e^+ \to (\pi^-\nu_\tau)(\pi^+\bar{\nu}_\tau)$, $e^-e^+ \to (((\pi^-\pi^0)\nu_\tau)(\rho^+\bar{\nu}_\tau) + ((\rho^-\nu_\tau)((\pi^+\pi^0)\bar{\nu}_\tau))$ and $e^-e^+ \to (((\pi^-\pi^0)\nu_\tau)(\pi^+\bar{\nu}_\tau) + (\pi^-\nu_\tau)((\pi^+\pi^0)\bar{\nu}_\tau))$. With $10^8\Upsilon$ one gets a sensitivity to g_V^b of $1.5 \cdot 10^{-2}$.

On the other hand, we can also combine the errors obtained in the $e^-e^+ \to (((\pi^-\pi^0)\nu_\tau)\tau^+ + \tau^-((\pi^+\pi^0)\bar{\nu}_\tau))$ and $e^-e^+ \to ((\pi^-\nu_\tau)\tau^+ + \tau^-(\pi^+\bar{\nu}_\tau))$ channels, and again with $10^8\Upsilon$ one gets a sensitivity of $1.3 \cdot 10^{-2}$.

Notice that our results have been obtained with a sample of $10^8\Upsilon$. For a different number of Υ produced, the sensitivity to g_V^b would simply re-scale by a factor $\sqrt{10^8/N_\Upsilon}$.

5 Conclusions

In this work we have studied the possibilities of a high luminosity B-meson facility to measure with high precision the $Z - b\bar{b}$ vector coupling. At the energies of $\Upsilon(1S)$ we have used the $\tau^-\tau^+$ channel to determine this coupling through the τ polarizations. A complete analysis of the hadronic decay modes of the τ lepton has been done, with special attention to the $\tau^- \to \pi^-\nu_\tau$ and $\tau^- \to \rho^-\nu_\tau$ as polarization analyzers. We have calculated the complete correlated cross section with the decays of both τs and from here the ideal statistical errors obtainable in the measurement of g_V^b. We have found that in one year run of a B-meson facility with $10^8\Upsilon$, one can get a sensitivity of $1.3 \cdot 10^{-2}$, comparable with the present precision in this coupling from the LEP/SLC measurements of R_b and A_b.

Acknowledgements. O.V. acknowledges the Generalitat Valenciana for a research fellowship. This work has been supported by Grant AEN96-1718 of the Spanish CICYT.

Appendix A

In a decay, $A(\sigma) \to B(\lambda_b) + C(\lambda_c)$, we can obtain the density matrix, ρ^{out}, describing the complete final state of particles B and C in terms of helicity amplitudes and the initial density matrix in the following way,

$$\rho^{out}_{\lambda,\lambda'} = \sum_{\sigma\sigma'} f_{\lambda,\sigma}\ (\Omega_1)\ \rho^{in}_{\sigma\sigma'}(\Omega)\ f^*_{\lambda',\sigma'}\ (\Omega_1) \tag{1}$$

where $\lambda = (\lambda_b, \lambda_c)$.

It is very convenient to express our initial density matrix in a basis of irreducible tensor operators, $T_{L,M}$, with coefficients $t_{L,M}$, [4,7],

$$\frac{\rho^{in}}{Tr(\rho^{in})} = \frac{1}{2j+1} \sum_{L,M}^{2j} (2L+1) t^{(a)\,*}_{L,M} T^{(a)}_{L,M} \tag{2}$$

The coefficients $t^{(a)}_{L,M}$ are the so-called multipole parameters,

$$\begin{aligned} Tr(\rho^{in}) t^{(a)}_{L,M}(\theta) &= Tr(\rho^{in} T^{(a)}_{L,M}) \\ &= \sum_{\sigma,\sigma'} (\rho^{out})_{\sigma,\sigma'} C(1L1|\sigma M\sigma') \end{aligned} \tag{3}$$

where $C(jLj|m'Mm)$ are Clebsch-Gordan coefficients.

For $L = 1$, we can relate [7] the usual polarizations and the multipole parameters,

$$P_{x'} = -(t_{1,1} - t_{1,-1}) = -2\ Re[t_{1,1}] \tag{4}$$

$$P_{y'} = i(t_{1,1} + t_{1,-1}) = -2\ Im[t_{1,1}] \tag{5}$$

$$P_{z'} = \sqrt{2}\ t_{1,0} \tag{6}$$

Using (2) to express the initial density matrix in the basis of irreducible tensors and replacing the helicity amplitudes in the C.M. frame of the decaying particle with (27), we get the final density matrix in terms of the multipole parameters of the A particle and the reduced helicity amplitudes,

$$\begin{aligned} \rho^{out}_{\lambda,\lambda'} = Tr(\rho^{in}) \frac{\sqrt{2j+1}}{4\pi} & \\ \times \sum_{L,M} \sqrt{2L+1}\ (-)^{j-\lambda'}\ T_\lambda T^*_{\lambda'} & \\ \times t^{(a)\,*}_{L,M} C(j,j,L|\lambda',-\lambda,\lambda'-\lambda) & \\ \times \mathcal{D}^{(L)*}_{M,\lambda-\lambda'}(\Omega) & \end{aligned} \tag{7}$$

This density matrix contains all the information available in the process. For instance, if we want the angular distribution we just have to take the trace on λ.Similarly the density matrix for one of the final particles is obtained taking the trace on the helicities of the other particle.

To define a complete set of observables we generalize (2)

$$\begin{aligned} \frac{\rho^{out}}{Tr(\rho^{out})} = \frac{1}{(2j_1+1)(2j_2+1)} & \\ \times \sum_{L,M,L',M'}^{2j} (2L+1)(2L'+1) & \\ \times \mathcal{C}^*_{L,M,L',M'} T^{(b)}_{L,M} T^{(c)}_{L',M'} & \end{aligned} \tag{8}$$

and then these generalized multipole parameters,$\mathcal{C}_{L,M,L',M'}$, include all the information on the density matrices of particles B and C, $t^{(b)}_{L,M} = \mathcal{C}_{L,M,0,0}$, $t^{(c)}_{L,M} = \mathcal{C}_{0,0,L,M}$ and additionally the correlations between them.

It is very important to notice here that (7) is only valid in the C.M. frame of the decaying particle. However, in general in the LAB frame the decaying particle will move with a momentum different from zero. So, we will be interested in the transformation properties of these density matrices under Lorentz boosts.

The transformation of an helicity amplitude under a boost from the C.M. frame to the LAB frame is just a rotation, the so-called Wigner rotation [12,16,17], that if we choose $\phi = 0$, is [7],

$$f^{(CM)}_{\lambda_b\lambda_c,\sigma}(\theta,\phi=0) = \sum_{\lambda'_b\lambda'_c}(-)^{\lambda'_c-\lambda_c}\, d^{j_b}_{\lambda'_b\lambda_b}(\omega_b) \times d^{j_c}_{\lambda'_c\lambda_c}(\omega_c)\, f^{(LAB)}_{\lambda'_b\lambda'_c,\sigma}(\theta,\phi=0) \quad (9)$$

The rotation of angle ω_i is given by [7],

$$\sin\omega_i = \frac{M_i\sinh\kappa\sin\theta}{\mid \mathbf{p}'_i\mid} \quad , \quad \cos\omega_i = \frac{E'_iE_i - \cosh\kappa M_i^2}{\mid \mathbf{p}_i \mid\mid \mathbf{p}'_i\mid} \quad (10)$$

where κ is the parameter of the boost from the C.M. frame of the decaying particle to the LAB frame, related to the velocity by $v = \tanh\kappa$, M_i the mass of the particle and $(E_i, \mathbf{p_i}(\theta))$ its four-momentum in C.M. of the decaying particle, transformed under the boost as $p'_i = \Lambda p_i$, in a frame where the boost is along the z-axis. From (10) we can see that under a boost collinear to the particle three-momentum, $\theta = 0$, our states do not suffer any rotation.

The helicity amplitudes in (9) are functions of two variables. For instance, we could choose the invariant variables (s,t,u), keeping the same expression in any frame. Nevertheless, as we have seen in (27), these helicity amplitudes have a specially simple form in terms of C.M. variables. Then, we have used this freedom to express, both the C.M. and the LAB helicity amplitudes in (9) in terms of C.M. variables.

As can see in (1), density matrices are a product of two helicity amplitudes, this means that using (9) we can get the transformation of the density matrix of particle B

$$\rho^{LAB} = d^{j_b}(\omega_b)\cdot\rho^{CM}\cdot d^{j_b\ T}(\omega_b) \quad (11)$$

It is very interesting to see how the multipole parameters are affected by these rotations. We use (3) to express ρ^{CM} in terms of these multipole parameters

$$\begin{aligned}\rho^{LAB}_{\sigma\sigma'} &= \frac{1}{2j+1}\sum_{LM}(2L+1) \\ &\quad\times\sum_{\lambda\lambda'} d^{(j)}_{\sigma\lambda}(\omega_a)(jLj|\lambda' M\lambda)d^{(j)\ T}_{\sigma'\lambda'}(\omega_a)t^{(b)\ *}_{LM} \\ &= \frac{1}{2j+1}\sum_{LM}(2L+1) \\ &\quad\times(jLj|\sigma'\sigma-\sigma')d^{(L)\ T}_{\sigma-\sigma' M}(\omega)t^{(b)\ *}_{LM}\end{aligned} \quad (12)$$

Comparing again with (3) we see that in LAB frame we get a new set of multipole parameters that are obtained simply by applying the rotation to the C.M. ones.

$$t^{(b)\ *}_{LM} = \sum_{M'} d^L_{MM'}(\omega_b)t^{(b)\ *}_{LM'} \quad (13)$$

This result is enough to obtain the multipole parameters and density matrices in the LAB frame.

Appendix B

In this appendix, we present the general method to calculate reduced helicity amplitudes, and we apply it to some examples in the $e^+e^- \to \tau^+\tau^-$ processes.

Reduced helicity amplitudes are easily calculable by means of (3) from the helicity amplitudes. So, our first step is to obtain these helicity amplitudes from the Feynman amplitudes that are calculated from the diagrams with Feynman rules. Taking into account the normalization we have defined for our reduced helicity amplitudes in (3) and (27) the difference between them and the Feynman amplitudes will just be a q^2-dependent phase space factor, irrelevant in all our observables. So, we can simply define:

$$M_{\sigma,\lambda_+,\lambda_-}(\theta) = K f_{\sigma,\lambda}(\theta) \quad (1)$$

with $M_{\sigma,\lambda_+,\lambda_-}(\theta)$ the Feynman amplitudes.

Now we will explicitly apply this procedure to the calculation of the reduced helicity amplitudes $T_{(\lambda,\lambda'),\xi}(\gamma)$ and $T_{(\lambda,\lambda'),\xi}(\gamma Z_A)$ corresponding to diagrams (1) + (2) and (3) + (5) in Fig. 1.

The kinematics in the C.M. frame of the e^+e^- system is defined by

$$\begin{aligned} l^\mu_- &= (E,0,0,|\mathbf{l}|) \qquad q^\mu = (l_- + l_+)^\mu \\ l^\mu_+ &= (E,0,0,-|\mathbf{l}|)\ k^\mu = (k^0, |\mathbf{k}|\sin\theta, 0, |\mathbf{k}|\cos\theta)\end{aligned} \quad (2)$$

where $l^\mu_\pm$ is the four-momentum of the $e^\pm$, whose helicities are $\xi_\pm = \pm 1/2$, k^μ the four-momentum of the τ^- and the helicities of the $\tau^\pm$ will be denoted as $\lambda_\pm = \pm 1/2$.

The Feynman amplitudes corresponding to these diagrams are

$$\begin{aligned} M^\gamma_{\lambda_-,\lambda_+,\xi_+,\xi_-}(\theta) &= i\frac{e^2}{s}(1+\frac{e^2}{s}Q_b^2|F_\Upsilon(q^2)|^2P_\Upsilon(q^2)) \\ &\quad\times V^v_\tau(\lambda_-,\lambda_+,\theta)g_{\nu\mu}V^{\mu *}_e(\xi_-,\xi_+)\end{aligned} \quad (3)$$

$$\begin{aligned} M^{\gamma Z_A}_{\lambda_-,\lambda_+,\xi_-,\xi_+}(\theta) &= i\frac{8G_F}{\sqrt{2}}g_A \\ &\quad\times(\frac{e^2}{s}Q_b g^b_V|F_\Upsilon(q^2)|^2P_\Upsilon(q^2) - g_V) \\ &\quad A^v_\tau(\lambda_-,\lambda_+,\theta)g_{\nu\mu}V^{\mu *}_e(\xi_-,\xi_+)\end{aligned} \quad (4)$$

where we have followed the notation of Sect. 2 and we have introduced the matrix elements of the leptonic currents,

$$\begin{aligned} V^\mu_l(\lambda_-,\lambda_+) &= \bar{u}(p_-,\lambda_-)\gamma^\mu v(p_+,\lambda_+) \\ A^\mu_l(\lambda_-,\lambda_+) &= \bar{u}(p_-,\lambda_-)\gamma^\mu\gamma_5 v(p_+,\lambda_+)\end{aligned} \quad (5)$$

Now we need to obtain an explicit expression for matrix elements of the currents. To do this, we follow the method of reference [18], which permits the calculation of these amplitudes using standard trace techniques. Then, the complete results for the vector and axial currents, in the CM frame and with the momenta along the z-axis are,

$$\begin{aligned} V^\mu_l(\lambda_-,\lambda_+=\lambda_-) &= (0,0,0,-2M_l) \\ V^\mu_l(\lambda_-,\lambda_+=-\lambda_-) &= (0,4E_p\lambda_-,-2E_p i,0) \\ A^\mu_l(\lambda_-,\lambda_+=\lambda_-) &= (-4M_l\lambda_-,0,0,0) \\ A^\mu_l(\lambda_-,\lambda_+=-\lambda_-) &= (0,2p,-4ip\lambda_-,0)\end{aligned} \quad (6)$$

We have obtained all leptonic currents needed, because they are well behaved Lorentz vectors or axial-vectors. Then, we just have to rotate them, if the momentum is in a different direction.

With all these results, we simply use (1) and (3) with our Feynman amplitudes, (3) and (4), to obtain the reduced helicity amplitudes. For instance, the expression for $M^{\gamma}_{\lambda=(1/2,1/2),\xi=(1/2,-1/2)}(\theta)$ is,

$$\begin{aligned} M^{\gamma}_{(1/2,1/2),(1/2,-1/2)}(\theta) &= i\frac{e^2}{s}(1+\frac{e^2}{s}Q_b^2|F_\Upsilon(q^2)|^2P_\Upsilon(q^2)) \\ &\times(0,-2M_\tau\sin\theta,0,-2M_\tau\cos\theta)\cdot(0,-2\frac{\sqrt{s}}{2},-2i\frac{\sqrt{s}}{2},0)^T \\ = i\frac{e^2}{s}(1+\frac{e^2}{s}Q_b^2|F_\Upsilon(q^2)|^2P_\Upsilon(q^2)) \\ &\times 4M_\tau\frac{\sqrt{s}}{2}\,(-\sqrt{2})\,(\frac{-\sin\theta}{\sqrt{2}}) \end{aligned} \tag{7}$$

where we have applied a rotation to the leptonic current of the τ with respect to (6) and we have taken the complex conjugate of (6) to obtain the electron current. The extra minus sign is due to the metric $g_{\mu\nu}$. In (7) we just have to remove the rotation matrix element $d_{10}(\theta)$, which is exactly the last term in this equation. This procedure has to be repeated with all the amplitudes and then, finally we get the following results

$$\begin{aligned} KT_{(+,+),1}(\gamma) &= -i4\sqrt{2}\,\frac{e^2}{s} \\ &\times(1+\frac{e^2}{s}Q_b^2|F_\Upsilon|^2P_\Upsilon)\,M_\tau\,\frac{\sqrt{s}}{2} \end{aligned} \tag{8}$$

$$\begin{aligned} KT_{(+,-),1}(\gamma) &= -i8\,\frac{e^2}{s} \\ &\times(1+\frac{e^2}{s}Q_b^2|F_\Upsilon|^2P_\Upsilon)\,p^0\,\frac{\sqrt{s}}{2} \end{aligned} \tag{9}$$

$$KT_{(+,+),1}(\gamma Z_A) = 0 \tag{10}$$

$$\begin{aligned} KT_{(+,-),1}(\gamma Z_A) &= -i8\,\frac{8G_F}{\sqrt{2}}\,g_A \\ &\times(\frac{e^2}{s}Q_b g_V^b|F_\Upsilon|^2P_\Upsilon - g_V) \\ &\times|\mathbf{p}|\,\frac{\sqrt{s}}{2} \end{aligned} \tag{11}$$

Similarly, we can obtain the reduced helicity amplitudes in (8).

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